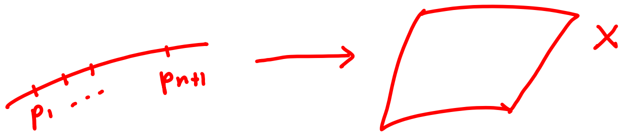
Cox Katz: Ch 10

Quantum Diff Eqt.

§ Gravitional Correlators

$\beta \in H_2(X, \mathbb{Z})$

$f :$ 

$$\begin{tikzcd} & X \quad f(p_i) \\ \overline{\mathcal{M}}_{g,n+1}(X,\beta) \arrow[ur, "e_i"] \arrow[r, "\pi_{n+1}"'] & \overline{\mathcal{M}}_{g,n}(X,\beta) \text{ forget } p_{n+1} \end{tikzcd}$$

Fibers of π_{n+1} ? $\quad \pi_{n+1}(f) = C / Aut(f)$

Roughly $\quad \overline{\mathcal{M}}_{g,n+1}(X,\beta) \xrightarrow{\pi_{n+1}} \overline{\mathcal{M}}_{g,n}(X,\beta)$

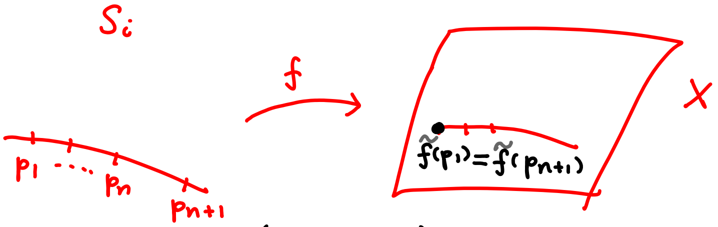
$\quad\quad\quad \wr$

$\quad\quad \overline{\mathcal{U}}_{g,n}(X,\beta)$ 'universal curve'

sections $\quad \begin{tikzcd} \overline{\mathcal{M}}_{g,n+1}(X,\beta) \arrow[r, "\pi_{n+1}"] & \overline{\mathcal{M}}_{g,n}(X,\beta) \arrow[l, bend left, "s_i"] \end{tikzcd}$

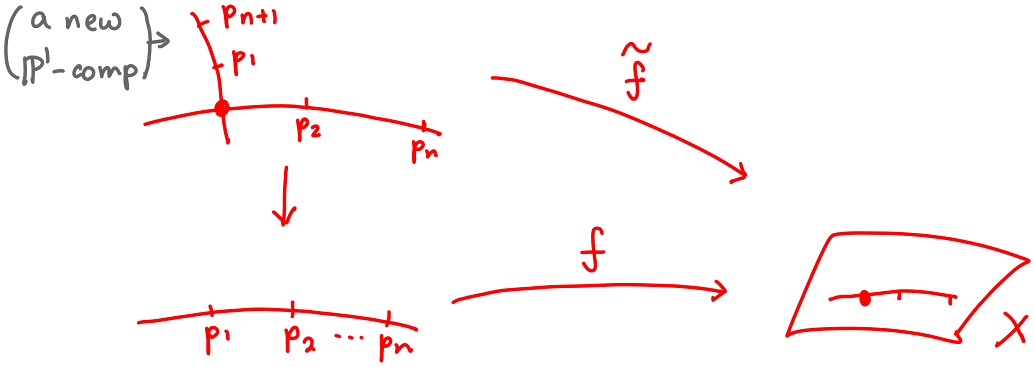
$s_i(f, p_1, \ldots, p_n)$

$= (\tilde{f}, p_1, \ldots, p_n, p_{n+1})$



(say $i=1$)

$\tilde{f}(p_1) = \tilde{f}(p_{n+1})$
violate def^{n} of stable maps.



Recall GW-inv. $\gamma_1, \dots, \gamma_n \in H^\bullet(X)$

$$\langle \gamma_1, \dots, \gamma_n \rangle_{g,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} e_1^*(\gamma_1) \wedge \cdots \wedge e_n^*(\gamma_n) \in \mathbb{Q}$$

or

$$\langle I_{g,n,\beta} \rangle : H^*(X, \mathbb{Q})^{\otimes n} \longrightarrow \mathbb{Q}$$

or

$$I_{g,n,\beta} : H^*(X,\mathbb{Q})^{\otimes n} \longrightarrow \underbrace{H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})}_{\text{couple w/ } c_1(L_1), \dots, c_1(L_n)}$$

Gravitational correlator

$$\langle \tau_{d_1}\gamma_1, \dots, \tau_{d_n}\gamma_n \rangle_{g,\beta} \triangleq \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} \left(c_1(L_1)^{d_1} e_1^*(\gamma_1)\right) \wedge \cdots \wedge \left(c_1(L_n)^{d_n} e_n^*(\gamma_n)\right)$$

$$\mathbb{C} \longrightarrow L_i \longrightarrow \overline{\mathcal{M}}_{g,n} \qquad L_i|_f = T^*_{p_i} C$$

cotangent line at the ith pt.

$$\langle \tau_{d_1}\gamma_1, \dots, \tau_{d_n}\gamma_n \rangle_{g,\beta} \triangleq \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} \left(c_1(L_1)^{d_1} e_1^*(\gamma_1)\right) \wedge \cdots \wedge \left(c_1(L_n)^{d_n} e_n^*(\gamma_n)\right)$$

$$\mathbb{C} \longrightarrow L_i \longrightarrow \overline{\mathcal{M}}_{g,n} \qquad L_i|_f = T^*_{p_i} C$$

Genus g coupling (anolog to Φ_g)

$$\langle\langle \tau_{d_1}\gamma_1, \dots, \tau_{d_n}\gamma_n \rangle\rangle_g \triangleq \sum_{k=0}^{\infty} \sum_{\beta} \frac{1}{k!} \langle \tau_{d_1}\gamma_1, \dots, \tau_{d_n}\gamma_n, (\gamma)^k \rangle_{g,\beta}\, q^\beta$$

$$\in \mathbb{C}[[t_0, \dots, t_m]]$$

with $\quad \gamma = \sum_j t_j T_j \in H^\bullet(X, \mathbb{Q}) \qquad q^\beta = e^{2\pi i \int_\beta \omega}$

Write $\quad \gamma = \sum_{d=0}^{\infty} \sum_j t_d^j \tau_d T_j$

$$\Phi_g^{grav}(\gamma) := \sum_{n=0}^{\infty} \sum_{\beta} \frac{1}{n!} \langle \gamma^n \rangle_{g,\beta}\, q^\beta$$

In particular

$$\langle\langle \tau_{d_1} T_{i_1}, \dots, \tau_{d_n} T_{i_n} \rangle\rangle_g = \left.\frac{\partial^n \Phi_g^{grav}}{\partial t_{d_1}^{i_1} \cdots \partial t_{d_n}^{i_n}}\right|_{t_d^j = 0 \text{ all}}$$

gravitational quantum product (coeff : $\mathbb{C}[[t_d^j]]$)

$$T_i *_g T_j = \sum_k \frac{\partial^3 \Phi_0^{grav}}{\partial t_i \partial t_j \partial t_k} T^k$$

$$T_i * T_j = \sum_k \underbrace{\frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_k}}_{\langle\langle T_i, T_j, T_k \rangle\rangle_0} T^k$$

usual big quantum product

Properties :

- $\langle \tau_{d_1}\gamma_1, \ldots, \tau_{d_n}\gamma_n \rangle_{g,\beta} \triangleq \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} (c_1(L_1)^{d_1} e_1^*(\gamma_1)) \wedge \cdots \wedge (c_1(L_n)^{d_n} e_n^*(\gamma_n))$

$= 0$ unless $\sum_{i=1}^n (2d_i + \deg \gamma_i) = 2(1-g)(\dim_{\mathbb{C}} X - 3) + 2\int_\beta c_1(X) + 2n$

- (Dilaton eqt) $\langle \tau_1, \tau_{d_1}\gamma_1, \ldots, \tau_{d_n}\gamma_n \rangle_{g,\beta} = (2g-2+n) \cdot \langle \tau_{d_1}\gamma_1, \ldots, \tau_{d_n}\gamma_n \rangle_{g,\beta}$

reason : $\underbrace{\overline{\mathcal{M}}_{g,n+1}(X,\beta)}_{\mathcal{C}_{g,n}(X,\beta)} \xrightarrow{\pi_1} \overline{\mathcal{M}}_{g,n}(X,\beta)$

univ. curve

$\mathcal{L}_1|_{\text{fiber}}$ $\nwarrow$ deg $2g-2+n$

$\|$

relative dualizing sheaf ($=0$ at p_i)

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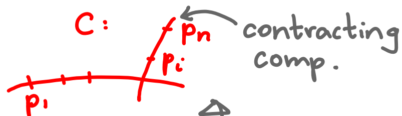
$$\begin{array}{ccc} \mathcal{L}_i & & \mathcal{L}_i' \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{g,n}(X,\beta) & \xrightarrow{\pi_n} & \overline{\mathcal{M}}_{g,n-1}(X,\beta) \end{array}$$

$(i<n)$

$\beta \neq 0$

$\mathcal{L}_i$ over $(f, C, p_1, \ldots, p_n) = T_{p_i}^* C$

$\Rightarrow$ $\mathcal{L}_i = \mathcal{L}_i'$ unless $(p_n = p_i)$



(*) $c_1(\mathcal{L}_i) = \pi_n^* c_1(\mathcal{L}_i') + \underbrace{\widetilde{D}_{(i,n|1,\ldots,\hat{i},\ldots,n-1)}}$

divisor of all such stable map.

$\Rightarrow$ •

$\langle \tau_{d_1}\gamma_1, \ldots, \tau_{d_{n-1}}\gamma_{n-1}, 1 \rangle_{g,\beta}$ (n)

$= \sum_{i=1}^{n-1} \langle \tau_{d_1}\gamma_1, \ldots, \underbrace{\cancel{\tau_{d_i}\gamma_i}}_{\tau_{d_i-1}\gamma_i}, \ldots, \tau_{d_{n-1}}\gamma_{n-1} \rangle_{g,\beta}$ ($n-1$)

- $\langle \tau_{d_1}\gamma_1, \ldots, \tau_{d_{n-1}}\gamma_{n-1}, D \rangle_{g,\beta}$ (n)

$= \langle \tau_{d_1}\gamma_1, \ldots, \tau_{d_{n-1}}\gamma_{n-1} \rangle_{g,\beta} \cdot \left(\int_\beta D\right)$

$+ \sum_{i=1}^{n-1} \langle \tau_{d_1}\gamma_1, \ldots, \underbrace{\cancel{\tau_{d_i}\gamma_i}}_{\tau_{d_i-1}\gamma_i \cup D}, \ldots, \tau_{d_{n-1}}\gamma_{n-1} \rangle_{g,\beta}$ ($n-1$)

for $\beta \neq 0$

- Splitting $\qquad \varphi : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$

$$\varphi^* I_{g,n,\beta}(\tau_{d_1}\gamma_1, \cdots, \tau_{d_n}\gamma_n)$$

$$= \sum_{\substack{i,\\ \beta=\beta_1+\beta_2}} I_{g_1,n_1+1,\beta}(\tau_{d_1}\gamma_1, \cdots, \tau_{d_{n_1}}\gamma_{n_1}, T_i) \otimes I_{g_2,n_2+1,\beta}(T^i, \tau_{d_{n_1+1}}\gamma_{n_1+1}, \cdots, \tau_{d_n}\gamma_n)$$

Remark : Virasoro Conjecture

$$Z := \exp\Big(\sum_{g=0}^{\infty} k^{2g-2} \Phi^g_{grav} \Big) \qquad \text{partition function}$$

$\exists$ explicit diffential operators (in t^j_d's)

$$L_{-1},\ L_0,\ L_1,\ L_2,\ \cdots\cdots \quad w/ \quad [L_n, L_m] = (n-m) L_{n+m}$$

s.t. $\quad L_n Z = 0$

When $X = pt \rightsquigarrow$ Witten Conj. / Kontsevich thm. ($\sim$ Matrix integral)

Vir conj. $\Rightarrow$ (TRR) topo. recursion relation

$$\tau_k \longrightarrow \tau_{k-1}$$

§ Givental connection

$$\begin{tikzcd} \mathcal{L}_1 \arrow[d] & \mathcal{L}_1' \arrow[d] \\ \overline{\mathcal{M}}_{0,n}(X,\beta) \arrow[r, "\pi"] & \overline{\mathcal{M}}_{0,3} \end{tikzcd} = * \qquad \mathcal{L}_1' : \text{trivial}$$

$$(f, C, p_1, \cdots, p_n) \longmapsto (C, p_1, p_2, p_3)$$

$$\Rightarrow \quad \mathcal{L}_1 = \cancel{\pi^* \mathcal{L}_1'} + \sum_{K \cup L = \{4,\dots,n\}} D_{(1,K \mid 2,3,L)}$$

$c_1(\mathcal{L}_1) \cup \tau_{d_1}$

$$\langle \tau_{d_1+1} T_{j_1},\ \tau_{d_2} T_{j_2},\ \tau_{d_3} T_{j_3},\ \cdots\cdots,\ \tau_{d_n} T_{j_n} \rangle_{0,\beta}$$

$\underline{\underline{\text{split}}}$

$$\sum_{\substack{K \cup L = \{4,\dots,n\} \\ \beta = \beta_1 + \beta_2}} \sum_a \pm \langle \tau_{d_1} T_{j_1},\ \tau_{d_{k_1}} T_{j_{k_1}}, \cdots, T_a \rangle_{0,\beta} \times \langle T^a,\ \tau_{d_2} T_{j_2},\ \tau_{d_3} T_{j_3},\ \tau_{d_{l_1}} T_{j_{l_1}}, \cdots \rangle_{0,\beta}$$

$$\langle\langle \tau_{d_1}\gamma_1, \cdots, \tau_{d_n}\gamma_n \rangle\rangle_0 \triangleq \sum_{k=0} \sum_{\beta} \frac{1}{k!} \langle \tau_{d_1}\gamma_1, \cdots, \tau_{d_n}\gamma_n, (\gamma)^k \rangle_{0,\beta}\, q^{\beta}$$

$$\Rightarrow \langle\langle \tau_{d_1+1} T_{j_1}, \tau_{d_2} T_{j_2}, \tau_{d_3} T_{j_3} \rangle\rangle_0 = \sum_a \langle\langle \tau_{d_1} T_{j_1}, T_a \rangle\rangle_0 \ \langle\langle T^a, \tau_{d_2} T_{j_2}, \tau_{d_3} T_{j_3} \rangle\rangle_0$$

(TRR)

$$\langle\langle \tau_{d_1+1}T_{j_1}, \tau_{d_2}T_{j_2}, \tau_{d_3}T_{j_3} \rangle\rangle_0 = \sum_a \langle\langle \tau_{d_1}T_{j_1}, T_a \rangle\rangle_0 \, \langle\langle T^a, \tau_{d_2}T_{j_2}, \tau_{d_3}T_{j_3} \rangle\rangle_0$$

Define
$$S_a \triangleq T_a + \sum_{n=0}^{\infty} \frac{1}{\hbar^{n+1}} \sum_j \langle\langle \tau_n T_a, T_j \rangle\rangle_0 T^j$$
$$= T_a + \sum_j \langle\langle \frac{T_a}{\hbar - c_1(\mathcal{L}_1)}, T_j \rangle\rangle_0 T^j$$

$$\nabla^g_{\frac{\partial}{\partial t_i}} S_a \triangleq \left(\hbar \frac{\partial}{\partial t_i} - T_i \overset{\text{Big}}{*}\right) S_a = 0$$

Givental "connection".

Pf: $$\hbar \frac{\partial S_a}{\partial t_i} = \sum_{n=0}^{\infty} \frac{1}{\hbar^n} \sum_j \langle\langle T_i, \tau_n T_a, T_j \rangle\rangle_0 T^j$$

$$= \underbrace{\sum_j \langle\langle T_i, T_a, T_j \rangle\rangle_0 T^j}_{n=0} + \underbrace{\sum_{n=0}^{\infty} \frac{1}{\hbar^{n+1}} \sum_j \langle\langle T_i, \tau_{n+1} T_a, T_j \rangle\rangle_0 T^j}_{\sum_{n=0}^{\infty} \frac{1}{\hbar^{n+1}} \sum_{j,c} \langle\langle \tau_n T_a, T_c \rangle\rangle_0 \langle\langle T_i, T^c, T_j \rangle\rangle_0 T^j}$$

shift $n \to n-1$ — use above TRR

$$= T_i * \underbrace{\left(T_a + \sum_{n=0}^{\infty} \frac{1}{\hbar^{n+1}} \sum_j \langle\langle \tau_n T_a, T_j \rangle\rangle_0 T^j\right)}_{S_a}$$

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- $\hbar \frac{\partial S_a}{\partial t_i} = T_i * S_a$ & $T_0 = 1 \Rightarrow \hbar \frac{\partial S_a}{\partial t_0} = S_a$

$\Rightarrow S_a = e^{t_0/\hbar} \times (\nexists\ t_0\text{-term})$

- Restrict to $M = H^0(X) + H^2(X)$

$(*_{\text{Big}} = *_{\text{small}}$ if let $q^\beta = 1)$ $\quad \delta = \sum_{i=1}^{r} t_i T_i$

$$S_a \triangleq T_a + \sum_{n=0}^{\infty} \frac{1}{\hbar^{n+1}} \sum_j \underbrace{\langle\langle \tau_n T_a, T_j \rangle\rangle_0}$$

$\beta = 0$ | $\beta \neq 0$

$$\underbrace{\sum_{k=0}^{\infty} \frac{1}{k!} \langle \tau_n T_a, T_j, (\delta)^k \rangle_{0,0}}_{\sum_{k=0}^{\infty} \frac{1}{k!} \left(\int_X T_a \cup T_j \cup \delta^k\right) \delta_{k,n+1}} + \sum_{k=0}^{\infty} \sum_{\beta \neq 0} \frac{1}{k!} \underbrace{\langle \tau_n T_a, T_j, (\delta)^k \rangle_{0,\beta}}_{\sum_{\mu+\nu=k} \frac{k!}{\mu!\nu!} \left(\int_\beta \mu\right)^\mu \langle \tau_{n-\nu}(T_a \cup \delta^\nu), T_j \rangle_{0,\beta}} \quad (q^\beta = 1)$$

(Ex) — (when $\beta \neq 0$) [divisor eqt].

$$\langle \tau_{d_1}\gamma_1, \cdots, \tau_{d_{n-1}}\gamma_{n-1}, D \rangle_{g,\beta}$$
$$= \langle \tau_{d_1}\gamma_1, \cdots, \tau_{d_{n-1}}\gamma_{n-1} \rangle_{g,\beta} \cdot \left(\int_\beta D\right)$$
$$+ \sum_{i=1}^{n-1} \langle \tau_{d_1}\gamma_1, \ldots, \underbrace{\tau_{d_i}\gamma_i}_{\tau_{d_i-1}\gamma_i \cup D}, \ldots, \tau_{d_{n-1}}\gamma_{n-1} \rangle_{g,\beta}$$

$$\underbrace{}_{\frac{1}{(n+1)!} \int_X T_a \cup T_j \cup \delta^{n+1}} \qquad \underbrace{}_{\sum_{\beta \neq 0} \sum_{\nu=1}^{\infty} e^{\int_\beta \delta} \frac{1}{\nu!} \langle \tau_{n-\nu}(T_a \cup \delta^\nu), T_j \rangle_{0,\beta}}$$

$$S_a \triangleq T_a + \sum_{n=0}^{\infty} \frac{1}{\hbar^{n+1}} \sum_j \langle\langle \tau_n T_a, T_j \rangle\rangle_0 T^j = T_a + \sum_j \langle\langle \frac{T_a}{\hbar - c_1(\mathcal{L}_1)}, T_j \rangle\rangle_0 T^j$$

$$= T_a + \sum_{n=0}^{\infty} \frac{1}{\hbar^{n+1}} \frac{1}{(n+1)!} \sum_j \underbrace{\left(\int_X T_a \cup T_j \cup \delta^{n+1} \right) T^j}_{= T_a \cup \delta^{n+1}}$$

$$+ \sum_{\beta \neq 0} \sum_j e^{\int_\beta \delta} \underbrace{\left(\sum_{\substack{n \geq \nu \\ \left(\tau_{n-\nu}=0 \text{ if } n-\nu<0\right)}} \frac{1}{\hbar^{n+1}} \frac{1}{\nu!} \langle \tau_{\underbrace{n-\nu}_{\text{set } = k}} (T_a \cup \delta^\nu), T_j \rangle_{0,\beta} \right)}_{\sum_{k=0}^{\infty} \frac{1}{\hbar^{k+1}} \langle \tau_k (T_a \cup e^{\delta/\hbar}), T_j \rangle_{0,\beta}} T^j$$

$$\sum_{k=0}^{\infty} \frac{1}{\hbar^{k+1}} \langle \tau_k (T_a \cup e^{\delta/\hbar}), T_j \rangle_{0,\beta} = \langle \frac{T_a \cup e^{\delta/\hbar}}{\hbar - c_1(\mathcal{L}_1)}, T_j \rangle_{0,\beta}$$

i.e.

$$S_a = e^{t_0/\hbar}\left(e^{\delta/\hbar} \cup T_a + \sum_{\beta \neq 0} \sum_{j=0}^{m} q^\beta \langle \frac{e^{\delta/\hbar} \cup T_a}{\hbar - c_1(\mathcal{L}_1)}, T_j \rangle_{0,\beta} T^j \right)$$

$$S_a = e^{t_0/\hbar}\left(e^{\delta/\hbar} \cup T_a + \sum_{\beta \neq 0} \sum_{j=0}^{m} q^\beta \langle \frac{e^{\delta/\hbar} \cup T_a}{\hbar - c_1(\mathcal{L}_1)}, T_j \rangle_{0,\beta} T^j \right.$$

flat sections of $\nabla = \hbar \nabla^s$ (set $\hbar = -2\pi i$)

bdl. $H^\bullet(V, \mathbb{C})$ over $K_{\mathbb{C}}(V) \supseteq D_\sigma \simeq (\Delta^*)^r$

$t_j \qquad q_j = e^{t_j}$

$\Rightarrow$ [illegible] of ∇ : $\mathcal{T}_j(s(T)) = s(e^{-T_j} \cup T)$

& $N_j(s(T)) = -s(T_j \cup T)$ where $N_j := \log \mathcal{T}_j$

[$\because$ only effect : $e^{\delta/h} \mapsto e^{(\delta - \hbar T_j)/\hbar} = e^{\delta/h} \cup e^{-T_j}$, or $T \rightsquigarrow e^{-T_j} \cup T$.]

$\Rightarrow$ $\tilde{s}(T) \triangleq \exp\left(\frac{-1}{2\pi i} \sum_j \log(q_j) N_j \right) s(T)$ can. ext^{n}. to $(\Delta)^r$

$$= T + \sum_{\beta \neq 0} \sum_{j=0}^{m} q^\beta \langle \frac{T}{\hbar - c_1(\mathcal{L}_1)}, T_j \rangle_{0,\beta} T^j$$ (Easy exercise).

$= T$ + higher deg terms.

$\tilde{s}(T)(0) = T$, $N_j \curvearrowright \{\tilde{s}(T_a)'s\}$ same $\cup T_j \curvearrowright \{T_a's\}$.

§ Relations in $QH^\bullet$

∇^g flat 'conn' on trivial bdl $\quad M \times H^\bullet(X, \mathbb{C}) \longrightarrow \underset{\sim\sim}{M} \overset{H^0 + H^2}{\overset{\|}{}}$

$$\langle \alpha, \beta \rangle = \int_X \alpha \cup \beta$$

$$J := \sum_j \langle s_j, 1 \rangle T^j \qquad J(t_0, \cdots, t_m, \hbar) \in H^\bullet(X)$$

$$= 1 + \sum_{n=0}^{\infty} \sum_{a=0}^{m} \frac{1}{\hbar^{n+1}} \langle\langle \tau_n T_a, 1 \rangle\rangle_0 T^a$$

$$P(\hbar \tfrac{\partial}{\partial t}, e^t, \hbar) J = 0 \Rightarrow P(T, q, 0) = 0 \quad \text{in } \underset{\text{(small)}}{QH^\bullet(X)}$$

Pf:
$$PJ = P(\hbar \tfrac{\partial}{\partial t}, e^t, \hbar)\left(\sum_j \langle s_j, 1 \rangle T^j\right)$$
$$= \sum_j \left(P \langle s_j, 1 \rangle\right) T^j$$

$$PJ = 0 \Rightarrow P \langle s_j, 1 \rangle = 0 \qquad \forall j$$

$$P(\hbar \tfrac{\partial}{\partial t}, e^t, \hbar) \langle s_j, 1 \rangle = 0 \qquad \forall j$$

$$\nabla^g_{\frac{\partial}{\partial t_i}} = \hbar \frac{\partial}{\partial t_i} - T_i * \quad \& \quad \tilde{\nabla}^g_{\frac{\partial}{\partial t_i}} = \hbar \frac{\partial}{\partial t_i} + T_i *$$

$$\Rightarrow \hbar \frac{\partial}{\partial t_i} \langle G, H \rangle = \langle \nabla^g_{\frac{\partial}{\partial t_i}} G, H \rangle + \langle G, \tilde{\nabla}^g_{\frac{\partial}{\partial t_i}} H \rangle$$

Recall $\quad \nabla^g s_j = 0 \qquad \forall j$

$$\tilde{\nabla}^g_{\frac{\partial}{\partial t_i}} 1 = T_i \qquad \tilde{\nabla}^g_{\frac{\partial}{\partial t_j}} \tilde{\nabla}^g_{\frac{\partial}{\partial t_i}} 1 = \left(\hbar \frac{\partial}{\partial t_j} + T_j *\right) T_i = T_j * T_i + O(\hbar)$$

$$\tilde{\nabla}_{\frac{\partial}{\partial t_k}} \cdots \tilde{\nabla}^g_{\frac{\partial}{\partial t_j}} \tilde{\nabla}^g_{\frac{\partial}{\partial t_i}} 1 = T_k * \cdots * T_j * T_i + O(\hbar)$$

$$P \langle s_j, 1 \rangle = 0 \Rightarrow \langle s_j, \underbrace{P(T, q, 0)}_{\text{wrt } *} \rangle + O(\hbar) = 0$$

$$\Rightarrow \langle s_j, P(T, q, 0) \rangle = 0 \qquad (\because \text{true } \forall \hbar)$$

$$\Rightarrow P(T, q, 0) = 0 \qquad (\because s_j\text{'s span } H^\bullet)$$

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Exercise: Over $M = H^0 + H^2$ write $\delta = \Sigma t_i T_i$, $q^\beta = e^{\int_\beta \delta}$

$$J = e^{\frac{t_0+\delta}{\hbar}} \left(1 + \sum_{\beta\neq 0} \sum_{a=0}^{m} q^\beta \left\langle \frac{T_a}{\hbar - c_1(\mathcal{L}_1)}, 1 \right\rangle_{0,\beta} T^a \right)$$

$$= e^{\frac{t_0+\delta}{\hbar}} \left(1 + o(\hbar^{-1}) \right)$$

Hint: $J = \sum_j \langle S_j, 1 \rangle T^j$ & $S_a = e^{t_0/\hbar}\left(e^{\delta/\hbar} \cup T_a + \sum_{\beta\neq 0} \sum_{j=0}^{m} q^\beta \left\langle \frac{e^{\delta/\hbar} \cup T_a}{\hbar - c_1(\mathcal{L}_1)}, T_j \right\rangle_{0,\beta} T^j \right)$

replace basis of $H^\cdot$ from T_a to $e^{\delta/\hbar} \cup T_a$.

For 2^{nd} "=", $\because$ $\langle T_a, 1 \rangle_{0,\beta} = 0$.

Note: $\underbrace{\sum_{a=0}^{m} \left\langle \frac{T_a}{\hbar - c_1(\mathcal{L}_1)}, 1 \right\rangle_{0,\beta} T^a}$ $\qquad e_1 : \overline{\mathcal{M}}_{0,2}(X,\beta) \xrightarrow{\text{ev at } p_1} X$

$$\underbrace{\int_{[\overline{\mathcal{M}}_{0,2}(X,\beta)]^{virt}} \frac{e_1^*(T_a)}{\hbar - c_1}}_{PD^{-1}\, e_{1*}\left(\frac{1}{\hbar - c_1} \cap [\overline{\mathcal{M}}_{0,2}(X,\beta)]^{virt} \right)} = \int_X T_a \cup PD^{-1}\, e_{1*}\left(\frac{1}{\hbar - c_1} \cap [\overline{\mathcal{M}}_{0,2}(X,\beta)]^{virt} \right)$$

$(\because\ \Sigma(\int T_a \cup \varphi) T^a = \varphi)$.

Example: $\mathbb{C}\mathbb{P}^1$

$H^0 + H^2$, $T_0 = 1 \overset{\shortparallel}{=} T^1$, $H = T_1 \overset{\shortparallel}{=} T^0$; $\delta = t_1 H$, $\beta = d \in H^2$, $q^\beta = e^{dt_1}$

$$J = e^{\frac{t_0+\delta}{\hbar}} \left(1 + \sum_{\beta\neq 0} \sum_{a=0}^{m} q^\beta \left\langle \frac{T_a}{\hbar - c_1(\mathcal{L}_1)}, 1 \right\rangle_{0,\beta} T^a \right)$$

$$= e^{\frac{t_0+t_1H}{\hbar}} \left(1 + \sum_{d=1}^{\infty} q^d \left(\left\langle \frac{H}{\hbar - c_1(\mathcal{L}_1)}, 1 \right\rangle_{0,d} 1 + \left\langle \frac{1}{\hbar - c_1(\mathcal{L}_1)}, 1 \right\rangle_{0,d} H \right) \right)$$

$$\overset{\text{claim}}{=} e^{\frac{t_0+t_1H}{\hbar}} \left(1 + \sum_{d=1}^{\infty} \left(\frac{q}{\hbar^2} \right)^d \frac{1}{(d!)^2} \left(1 - 2\frac{H}{\hbar}\left(1 + \frac{1}{2} + \dots + \frac{1}{d}\right)\right)\right)$$

$$= e^{\frac{t_0+t_1H}{\hbar}} \sum_{d=0}^{\infty} q^d \left(d!\, \hbar^{d-1} \left(\sum_{j=1}^{d} \frac{1}{j} \right) H + d!\, \hbar^d \right)^{-2} \qquad (\because H^2 = 0)$$

$$= e^{\frac{t_0+t_1H}{\hbar}} \sum_{d=0}^{\infty} q^d \left((H+\hbar)(H+2\hbar) + \dots\dots + (H+d\hbar) \right)^{-2} \qquad (\because H^2 = 0)$$

Can check directly that $\left(\left(\hbar \frac{d}{dt_1} \right)^2 - e^{t_1} \right) J = 0$.

$\rightsquigarrow$ $H^2 - q = 0$ in $QH^*(\mathbb{P}^1)$.

Claim: For $\mathbb{CP}^1$.

(1) $\langle \tau_{2d-1} H, 1 \rangle_{0,d} = (d!)^{-2}$, $\langle \tau_{2d} H, 1 \rangle = 0$ wrong dim.

(2). $\langle \tau_{2d}, 1 \rangle_{0,d} = \frac{-2}{(d!)^2}\left(1 + \frac{1}{2} + \dots + \frac{1}{d}\right)$, $\langle \tau_{2d-1}, 1 \rangle = 0$

Pf:

$$\langle \tau_1 H, 1 \rangle_{0,1} = \langle H \rangle_{0,1} = \langle H, H, H \rangle_{0,1} = 1$$

$$\left(\because \langle \tau_{d_1}\gamma_1, \cdots, \tau_{d_{n-1}}\gamma_{n-1}, 1 \rangle_{g,\beta} = \sum_{i=1}^{n-1} \langle \tau_{d_1}\gamma_1, \dots, \underbrace{\cancel{\tau_{d_i}\gamma_i}}_{\tau_{d_i-1}\gamma_i}, \dots, \tau_{d_{n-1}}\gamma_{n-1} \rangle_{g,\beta} \right)$$

Fund. class eqt.

For $\langle \tau_{2d-1} H, 1 \rangle_{0,d}$:

$\overline{\mathcal{M}}_{0,5}(\mathbb{P}^1, d) \longrightarrow \overline{\mathcal{M}}_{0,4}$, $D(23|45) \sim D(25|34)$

$$\Rightarrow \int_{\overline{\mathcal{M}}_{0,5}(\mathbb{P}^1,d)} c_1(\mathcal{L}_1)^{2d-1} \cup e_1^*(H) \cup e_3^*(H) \cup e_4^*(H) \cup D(23|45)$$

$$= \int_{\overline{\mathcal{M}}_{0,5}(\mathbb{P}^1,d)} c_1(\mathcal{L}_1)^{2d-1} \cup e_1^*(H) \cup e_3^*(H) \cup e_4^*(H) \cup D(25|34)$$

split (many terms vanish)

$$\Rightarrow \underbrace{\langle \tau_{2d-1} H, 1, H, H \rangle_{0,d}}_{d^2 \langle \tau_{2d-1} H, 1 \rangle_{0,d}} \underbrace{\langle 1, H, 1 \rangle_{0,0}}_{1} = \underbrace{\langle \tau_{2d-1} H, 1, 1, 1 \rangle_{0,d-1}}_{\langle \tau_{2d-3} H, 1 \rangle_{0,d-1}} \underbrace{\langle H, H, H \rangle_{0,1}}_{1}$$

(divisor eqt twice) (fund. class eqt twice)

$$\Rightarrow \langle \tau_{2d-1} H, 1 \rangle_{0,d} = d^{-2} \langle \tau_{2d-3} H, 1 \rangle_{0,d-1} = (d!)^{-2} \underbrace{\langle \tau_1 H, 1 \rangle_{0,1}}_{1} = (d!)^{-2}$$

$$\langle \tau_2, 1 \rangle_{0,1} = \langle \tau_1 \rangle_{0,1} \quad \text{(Fund. class eqt.)}$$

$$= \int_{\mathbb{P}^1} c_1(T^*\mathbb{P}^1) \left(\because \begin{array}{ccc} \mathcal{L}_1 & \longleftrightarrow & T^*\mathbb{P}^1 \\ \overline{\mathcal{M}}_{0,1}(\mathbb{P}^1, 1) & \simeq & \mathbb{P}^1 \end{array} \right) = -2$$

Consider splitting (using $\cup\, c_1(\mathcal{L}_1)^{2d} \cup e_3^* H \cup e_4^* H$),

$$\underbrace{\langle \tau_{2d}, 1, H, H \rangle_{0,d}}_{\substack{d^2 \langle \tau_{2d}, 1 \rangle_{0,d} \\ + 2d \langle \tau_{2d-1} H, 1 \rangle_{0,d}}} \underbrace{\langle 1, H, 1 \rangle_{0,0}}_{1} = \underbrace{\langle \tau_{2d}, 1, 1, 1 \rangle_{0,d-1}}_{\langle \tau_{2d-2}, 1 \rangle_{0,d-1}} \underbrace{\langle H, H, H \rangle_{0,1}}_{1}$$

$\Rightarrow \langle \tau_{2d}, 1 \rangle_{0,d}$ ✓

#

Example: $\mathbb{CP}^n$.

small quantum product $H^{n+1} = e^{t_1}$

H^i $(i \le n)$ has no t_0, t_1 dependence

$$\Rightarrow \left(\tilde{\nabla}_{\frac{d}{dt_1}} \right)^{n+1} 1 = H^{n+1} \leftarrow \text{quantum product}$$

$$= e^{t_1} \qquad \left(\begin{array}{l} QH^* = \mathbb{Q}[H]/H^{n+1} - e^{t_1} \\ \text{proven before} \end{array} \right)$$

$$\overset{\text{Thm.}}{\Longleftrightarrow} \left(\left(\hbar \frac{d}{dt_1} \right)^{n+1} - e^{t_1} \right) J = 0$$

In fact,

$$J = e^{(t_0 + t_1 H)/\hbar} \sum_{d=0}^{\infty} e^{d t_1} \left((H+\hbar)(H+2\hbar) \cdots (H+d\hbar) \right)^{-(n+1)}$$

(can be proved by localization method.)

Example: V CY^3

H^0	H^2	H^4	H^6	
$T_0 = 1$	$T_1, \cdots, T_r$	$T^1, \cdots, T^r$	T^0	$\delta = \sum_{i=1}^{r} t_i T_i$

$$J = e^{\frac{t_0 + \delta}{\hbar}} \left(1 + \sum_{\beta \ne 0} \sum_{a=0}^{m} q^\beta \left\langle \frac{T_a}{\hbar - c_1(\mathcal{L}_1)}, 1 \right\rangle_{0,\beta} T^a \right) \qquad \left(\begin{array}{c} \text{general} \\ \text{formula} \end{array} \right)$$

$$= e^{\frac{t_0 + \delta}{\hbar}} \left(1 + \sum_{\beta \ne 0} q^\beta \left(\hbar^{-2} \sum_{a=1}^{r} \underbrace{\underbrace{\langle \tau_1 T_a, 1 \rangle_{0,\beta}}_{\langle T_a \rangle_{0,\beta}}}_{(\int_\beta T_a) N_\beta} T^a + \hbar^{-3} \underbrace{\underbrace{\langle \tau_2, 1 \rangle_{0,\beta}}_{\langle \tau_1 \rangle_{0,\beta}}}_{-2 N_\beta} T^0 \right) \right)$$

(Fund. class eqt.)

(dilaton eqt)

Dilaton eqt: $\langle \tau_1, \tau_{d_1}\gamma_1, \cdots, \gamma_{d_n}\gamma_n \rangle_{g,\beta} = (2g-2+n) \cdot \langle \tau_{d_1}\gamma_1, \cdots, \gamma_{d_n}\gamma_n \rangle_{g,\beta}$

$$J = e^{\frac{t_0 + \delta}{\hbar}} \left(1 + \sum_{\beta \ne 0} q^\beta \left(\hbar^{-2} N_\beta \underbrace{\left(\sum_{a=1}^{r} \int_\beta T_a \right) T^a}_{\beta \in H_2 \simeq H^4} - 2 \hbar^{-3} N_\beta T^0 \right) \right)$$

$$J = e^{\frac{t_0+\delta}{\hbar}} \left(1 + \hbar^{-2} \sum_{\beta\neq 0} N_\beta q^\beta \beta - 2\hbar^{-3} \sum_{\beta\neq 0} N_\beta q^\beta \, pt. \right)$$

Recall: $\Phi = \frac{1}{6} \int_V \delta^3 + \sum_{\beta\neq 0} N_\beta q^\beta$

(Ex)

$$\Rightarrow J = e^{t_0/\hbar} \left(1 + \hbar^{-1} \sum_{a=1}^{r} t_a T_a + \hbar^{-2} \sum_{a=1}^{r} \frac{\partial \Phi}{\partial t_a} T^a + \hbar^{-3} \left(\sum_{a=1}^{r} t_a \frac{\partial \Phi}{\partial t_a} - 2\Phi \right) T^0 \right)$$

$$\Rightarrow \frac{\partial^2 J}{\partial t_i \partial t_j} = \hbar^{-2} e^{(t_0+\delta)/\hbar} \sum_a \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_a} T^a$$

$$= \hbar^{-2} e^{(t_0+\delta)/\hbar} T_i * T_j$$

Say $b_2 = 1$ $\quad H := T_1$, $\quad C := T^1$. $\quad (q = e^{t_1})$

$$H * H = \underbrace{\langle H, H, H \rangle}_{Y(q)} C$$

Above $\Rightarrow$ $\left(\hbar \frac{d}{dt_1}\right)^2 J = e^{(t_0 + t_1 H)/\hbar} H * H$

$$\Rightarrow \left(\hbar \frac{d}{dt_1}\right)^2 \left(\frac{1}{Y(q)} \left(\hbar \frac{d}{dt_1}\right)^2 \right) J = e^{(t_0+t_1 H)/\hbar} H \cup H \cup C = 0 \qquad (\because \in H^8)$$

$$\Rightarrow H * H * (H * H / Y) = 0 \quad \text{in} \quad QH^*(V)$$

Thm: V : CY^3

$P_\nabla = \sum_\alpha A_\alpha(q) \nabla^\alpha$ $\qquad a = (a_1, \dots, a_r)$ multi-index

write $P(\hbar \frac{\partial}{\partial t}, e^t, \hbar) := \sum_\alpha \left(\frac{\hbar}{2\pi i}\right)^{m - |\alpha|} A_\alpha(e^t) \left(\hbar \frac{\partial}{\partial t}\right)^\alpha$ $\qquad \hbar$-homog. of P_∇

where $m = \max_\alpha \{ |\alpha| : A_\alpha(q) \neq 0 \}$ order of P_∇.

$$P_\nabla 1 = 0 \iff P(\hbar \tfrac{\partial}{\partial t}, e^t, \hbar) J = 0$$

(~Picard-Fuch eqt)

$$\Longrightarrow P_m(T, q) = 0 \quad \text{in} \quad QH^\cdot(V)$$